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# The backbone and conductivity of random clusters

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**Abstract.** We study the properties of the backbone of, and conduction in, random clusters which have homogeneous interior structure. These include the largest percolation cluster at  $p_c$ , lattice animals and the Witten-Sander (ws) aggregates. For lattice animals the fractal dimension  $\bar{d}_a$  of the backbone is estimated, for the first time, and is found to be about 1.14 at  $d = 2$  and about 1.39 at  $d = 3$ . These values are much lower than the corresponding values for the animals themselves. The fractal and spectral dimensions of the backbone of the ws aggregates are estimated to be about 1.25 and 1.06 at  $d = 2$ , respectively. A recent hypothesis which relates the fractal dimension of random walks on these clusters to those of the clusters and their backbones is also discussed. It is shown that for the ws aggregates  $d_s = \bar{d}_s = 2$  on a Bethe lattice (i.e. at  $d = \infty$ ), in contrast with  $d_s = 4/3$  and  $\bar{d}_s = 1$  for percolation clusters and lattice animals. The conductivity exponent  $t_a$  of lattice animals is found to be  $t_a(d = 2) \approx 0.73$  and  $t_a(d = 3) \approx 1.19$  and  $t_a(d \geq 8) = 2$ , whereas for the ws aggregates we find  $t_{ws}(d = 2) \approx 0.67$  and  $t_{ws}(d = 3) \approx 0.94$  and  $t_{ws}(d \rightarrow \infty) = 1$ .

## 1. Introduction

Diffusion on fractal structures has been the subject of considerable recent interest. De Gennes (1976) introduced this problem in an attempt to probe the structure of percolation clusters (for a review, see Mitescu and Roussenoq 1983). In this problem, one considers the motion of a particle ('the ant') which performs an unbiased, nearest-neighbour (Pólya) random walk on a percolation network ('the labyrinth'). For such a walk, the root-mean-squared displacement  $R$  of the random walk is related to the number of steps  $N_w$  of the walk through the relation  $R \sim N_w^{1/d_w}$ , where  $d_w$  is the fractal dimension of the walk. If  $R$  is much larger than  $\xi_p$ , the percolation correlation length, the fractal dimension  $d_w$  takes on its superuniversal value of 2 at all dimensions. But if  $R \ll \xi_p$ , then  $d_w$  depends on the dimensionality of the system. In the latter case the random walk takes place on either the largest percolation cluster at the percolation threshold  $p_c$  (in which case  $\xi_p = \infty$ ) or slightly above  $p_c$  and at short times. The largest percolation cluster at  $p_c$  is recognised to be a fractal object and is characterised by a fractal dimension  $d_p$ . This fractal dimension is related to other percolation exponents at dimension  $d$  through (Kirkpatrick 1978, Stauffer 1979)  $d_p = d - \beta/\nu$ , where  $\beta$  is the critical exponent of the percolation probability and  $\nu$  the critical exponent of the

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percolation correlation length. No exact values for  $d_w$  and  $d_p$  are known for  $2 \leq d < 6$ , although with the possibly exact values of  $\beta(d=2) = 5/36$  and  $\nu(d=2) = 4/3$  (see the review by Sahimi 1983) one obtains the possibly exact value  $d_p(d=2) = 91/48$ ; for  $d \geq 6$  one has  $d_p = 4$ .

It was established by Gefen *et al* (1983) that the value of  $d_w$  depends on whether one averages  $R$  over all percolation clusters or considers  $R$  for only a single cluster. In the former case  $d_w = 2(2\nu + t - \beta)/(2\nu - \beta)$ , whereas for the latter case and for the largest percolation cluster at  $p_c$ ,  $d_w = 2 + (t - \beta)/\nu$ , where  $t$  is the critical exponent of percolation conductivity. Thus for  $d \geq 6$  one has  $d_w = \infty$  for the former case, whereas  $d_w = 6$  for the latter one.

Alexander and Orbach (1982) defined a new fractal dimension  $d_s$  which they called the spectral (fracton) dimension. It is given by

$$d_s = 2d_p/d_w \quad (1)$$

and it can be thought of as a fractal dimension in reciprocal space. This definition is general and is applied to any fractal object with a random walk fractal dimension  $d_w$ . Based on numerical evidence, Alexander and Orbach conjectured that for the largest percolation cluster at  $p_c$  the spectral dimension  $d_s$  is a superuniversal quantity and equals  $4/3$  at all dimensions. The most recent and accurate estimate of  $t$  at  $d=2$  by Zabolitzky (1984), Lobb and Frank (1984), Herrmann *et al* (1984) and Hong *et al* (1984) yields  $d_s(d=2) \approx 1.321$ , which indicates rather strongly that the Alexander-Orbach (AO) conjecture may fail at  $d=2$ . Sahimi (1984a) and Aharony and Stauffer (1984) have argued that even if the AO conjecture is exact, it is only for fractal dimensionalities  $d_p \geq 2$  of the largest percolation cluster at  $p_c$ .

Several generalisations of the problem of 'the ant in the labyrinth' have recently been proposed and studied. In one of them, one studies the mixing of two miscible fluids in an unsaturated porous medium as the random walk of a tracer particle on the infinite percolation cluster. This problem differs in several important aspects from the original ant problem. There is an external field in one principal direction of the percolation network (the overall pressure gradient in the porous medium), and thus the ant (the tracer particle) favours more steps in the longitudinal direction (parallel to the overall pressure gradient or the external field). The local transition probabilities are proportional to the local fluxes in the bonds of the network and steps against the direction of local potential drop along a bond are forbidden. Monte Carlo simulations show that the overall motion of the ant is characterised by *two* diffusion coefficients, one for the longitudinal direction and another for the transverse directions (perpendicular to the longitudinal direction). This is because the external field induces local anisotropy in local fluxes. Usually, the longitudinal diffusion coefficient is at least one order of magnitude larger than the transverse one far from  $p_c$ . Moreover, it appears that as  $p_c$  is approached, the longitudinal diffusion coefficient *diverges*, whereas the transverse diffusion coefficient *vanishes*. Because the transition probabilities are proportional to the local fluxes within bonds of the network, the ant performs its walk only on the backbone of the cluster, since the dead-end part of the cluster does not carry any flow. The ant can diffuse into the dead-end part of the cluster with a molecular diffusion coefficient which is usually much smaller than its diffusion coefficients for the backbone of the cluster. This problem, dispersion in flow through a porous medium, has many applications in the fields of chemical and petroleum engineering. Monte Carlo simulation results and scaling laws for the diffusion coefficients of the ant in

this problem have been given elsewhere (Sahimi *et al* 1982, 1983, 1984a, b, Sahimi 1984c).

Wilke *et al* (1984), Sahimi and Jerauld (1984) and Havlin *et al* (1984) recently studied random walks on lattice animals in two, three and four dimensions. Meakin and Stanley (1983) studied random walks on the diffusion-limited aggregates first studied by Witten and Sander (1981). These aggregates are fractal objects which are distinctly different from percolation clusters and lattice animals (Gould *et al* 1983). Meakin and Stanley (1983) found that the spectral dimension of the Witten-Sander (ws) aggregates lies between 1.2 and 1.4. One can also study self avoiding walks (saw) on percolation clusters (Kremer 1981, Sahimi 1984b, Rammal *et al* 1984). This problem is interesting because the statistics of such saws might be related to those of the backbone of the clusters.

The largest percolation cluster at  $p_c$ , lattice animals and the ws aggregates are all examples of fractals with *homogeneous interior structure*. Moreover, the fractal dimensionality of these clusters is a monotonic function of dimensionality  $d$  (up to their upper critical dimensionality). Since diffusion and DC conduction are related to each other through the well known Einstein relation, and since conduction takes place only on the backbone, it is worthwhile investigating directly DC conduction and backbone properties on such fractals. The purpose of this paper is to provide the results of such a study. We calculate, for the first time, the spectral and fractal dimensions of the backbone of lattice animals and the ws aggregates from which one can obtain useful insight into the structure of these fractals. A scaling relation which relates the conductivity exponent of these fractals to their spectral and fractal dimensions is used to find the conductivity exponents of these fractals. This helps one to distinguish between these random clusters. This work complements our previous work on random walks on the backbone of percolation clusters mentioned above.

## 2. Percolation clusters

If resistors are placed between the nearest-neighbour sites of a random cluster of linear dimension  $L$ , then the *conductance*  $g(L)$  of the cluster scales with  $L$  as

$$g(L) \sim L^{\beta_L}, \quad (2)$$

which is similar to quantum conduction. Rammal and Toulouse (1983) proposed that for a fractal of spectral dimension  $d_s < 2$  and fractal dimension  $d_f$ ,  $\beta_L$  is given by

$$\beta_L = d_f(1 - 2/d_s); \quad (3)$$

for  $d \geq 6$  one has  $\beta_L = -2$ . By using the definition of  $d_s$  and the equations which relate  $d_p$  and  $d_w$  to exponents  $t$ ,  $\nu$  and  $\beta$  we obtain,  $\beta_L = d - 2 - t/\nu$ . Therefore, the exponent  $\beta_L$  is the same for percolation clusters and their backbone since  $\nu$  is the same for both of these clusters. Thus one may write  $\beta_L = \bar{d}_p(1 - 2/\bar{d}_s) = \beta_L$ , where  $\bar{d}_p$  is the fractal dimension of the backbone and  $\bar{d}_s = 2\bar{d}_p/\bar{d}_w$  is the spectral dimension of the backbone. The equality of  $\beta_L$  and  $\bar{\beta}_L$  was postulated by Puech and Rammal (1983) and was also discussed by Stanley and Coniglio (1984). Here  $\bar{d}_p = d - \beta_B/\nu$ , where  $\beta_B$  is the critical exponent of backbone fraction (Kirkpatrick 1978).  $\bar{d}_w$  is the fractal dimension of random walk *on the backbone*. It is straightforward to show that  $\bar{d}_w = 2 + (t - \beta_B)/\nu$  for the backbone of the largest percolation cluster at  $p_c$ . From the equality of  $\beta_L$  and

$\bar{\beta}_L$  one can calculate  $\bar{d}_s$ , as was done by Stanley and Coniglio (1984). One then finds that  $\bar{d}_s$  varies continuously from a value of about 1.25 at  $d = 2$  to  $\bar{d}_s = 1$  for  $d \geq 6$ , in contrast with  $d_s$  for the largest percolation cluster at  $p_c$  which remains approximately constant.

Wilke *et al* (1984) attempted to establish a connection between the fractal dimension of random walks on a cluster and those of the cluster and its backbone. By making several assumptions they proposed that

$$d_w = d_f + \bar{d}_f. \quad (4)$$

This relation was postulated to hold for percolation clusters as well as lattice animals. But the available data show that it does not hold for the percolation clusters. For percolation clusters loops play an important role and their removal changes the universality class. On the other hand, for fractals, such as lattice animals, loops are not important (Lubensky and Isaacson 1979) and their removal does not generate a new universality class. In this case equation (4) may be expected to hold. This has interesting consequences that will be discussed in the following sections.

If we assume that the usual scaling relations between the cluster exponents (Stauffer 1979) also hold for the critical exponents of the backbone, as the work of Harris (1983) indicates that this is indeed the case, then, following Gefen *et al* (1983), it can be shown that if one averages the root-mean-squared  $R_B$  of the random walk on the backbone over the distribution of the backbone of *all* clusters, one obtains

$$\bar{d}_w = 2(2\nu + t - \beta_B)/(2\nu - \beta_B). \quad (5)$$

Thus in this case  $\bar{d}_w$  remains finite for  $d \geq 6$ , whereas  $d_w = \infty$  for percolation clusters for the same situation. This non-divergence of  $\bar{d}_w$  for the percolation clusters for  $d \geq 6$  may mean that this averaging amounts to a series process in which any ant will eventually find itself in an isolated and finite backbone.

We note here that the exponent  $\bar{\beta}_L$  (and  $\beta_L$ ) for the backbone is the same as the exponent  $-\zeta/\nu$  of Stanley and Coniglio (1984).  $\zeta$  is defined by  $\zeta = t - (d-2)\nu$  and describes the resistance between two sites on the cluster which are separated by a distance comparable to  $\xi_p$ . However, their conjecture that  $\bar{d}_s = \zeta$ , fails in low dimensions, since from the equality of  $\beta_L$  and  $\bar{\beta}_L$ , the relation  $\beta_L = -\zeta/\nu$  and their conjecture one can obtain

$$\bar{d}_p = \nu\beta_L^2/(2 + \nu\beta_L). \quad (6)$$

By using the current values of  $\nu$ ,  $\beta$  and  $t$  we can calculate  $\beta_L$  and thus  $\bar{d}_p$  by invoking (6). We obtain  $\bar{d}_p(d=2) \approx 1.80$  and  $\bar{d}_p(d=3) \approx 1.63$ , which means that equation (6) predicts that  $\bar{d}_p$  is a non-monotonic function of dimensionality  $d$  and attains a minimum at about  $d = 3$ . Field-theoretic formulation of the backbone problem by Harris (1983) yields  $\bar{d}_p = 2 + \varepsilon/21 + \dots$  where  $\varepsilon = 6 - d$ . This does indicate that  $\bar{d}_p$  is a non-monotonic function of  $d$ , but it predicts that  $\bar{d}_p$  has a *maximum* at a dimensionality  $d_p^*$  such that  $4 < d^* < 5$ , whereas the conjecture of Stanley and Coniglio predicts that  $\bar{d}$  has a *minimum* at about  $d^* \approx 3$ . This non-monotonic dependence of  $\bar{d}_p$  on  $d$  may explain the failure of the AO conjecture for the backbone of percolation clusters.

### 3. Lattice animals

We now investigate the backbone and conduction properties of lattice animals; the ws aggregates will be studied in the next section. To define the backbone of these

fractals, we consider two sites  $i$  and  $j$  which are separated by a distance comparable to the radius of the fractals. Then the backbone sites (or bonds) are the ones that belong to at least two distinct self-avoiding walks between sites  $i$  and  $j$ . The rest are the dead-end or dangling ones. If  $R_B$  is the radius of the backbone, the fractal dimension of the backbone of the cluster is defined by

$$R_B \sim N_B^{1/\bar{d}_f}, \quad (7)$$

where  $N_B$  is the number of elements in the backbone and equation (7) is valid for the limit of  $N_B \rightarrow \infty$ .

We first obtain estimates of  $\bar{d}_a$  for the backbone of lattice animals in two and three dimensions. To do this, we employ a position space renormalisation group (PSRG) method. We consider *site* animals, i.e. connected clusters of sites originating from a single site. To construct a renormalisation group (RG) transformation of the backbone we first assign a fugacity  $S_B$  to each site in the backbone. We then determine the generating function

$$G(S_B, b) = \sum_i C_i S_B^i, \quad (8)$$

where  $C_i$  is the total number of backbones with  $i$  sites which span the RG cell in a particular direction on a cell of linear dimension  $b$ . The RG transformation is defined by requiring that the generating function for the spanning backbones is invariant on the original and rescaled levels. This leads us to an equation for the renormalised fugacity  $S'_B$  on the rescaled cell:

$$S'_B = G(S_B, b).$$

The fractal dimension of the backbone is given by  $\bar{d}_a = \ln \lambda_B / \ln b$ , where  $\lambda_B = (\partial S'_B / \partial S_B)_{S_B^*}$  is the eigenvalue of the linearised RG transformation.  $S_B^*$  is the fixed point of the transformation, i.e. the solution of the equation  $S_B^* = G(S_B^*, b)$ . This PSRG method is similar to the one developed by Family (1983) for the calculation of the fractal dimension of animals themselves.

The recursion relation for  $S'_B$  was determined for cells of size  $b = 2-4$  on a square lattice. In three dimensions it is not possible to calculate, in closed form, the recursion relation for  $S'_B$  for  $b > 2$ . The results for the square lattice are  $\bar{d}_a = 1.409, 1.317$  and  $1.273$  for  $b = 2, 3$  and  $4$ , respectively. By a similar procedure one finds  $d_a = 1.409, 1.431$  and  $1.447$  for the fractal dimension of the animal itself for cells of size  $b = 2, 3$  and  $4$ , respectively. These values extrapolate to  $d_a \approx 1.55$  and  $\bar{d}_a \approx 1.14$ . The estimate for  $d_a$  is only about 1% smaller than the most accurate estimate of  $d_a$  in two dimensions,  $d_a \approx 1.56$  (Derrida and DeSeze 1982).

In three dimensions the  $b = 2$  cell yields  $\bar{d}_a = 1.722$ . Let us define  $f(b) = \bar{d}_a / \bar{d}_a(b)$ , where  $\bar{d}_a$  is the correct value of this exponent and  $\bar{d}_a(b)$  is its value obtained from PSRG calculations with a cell of size  $b$ . In two dimensions we obtain  $f(2) \approx 1.14 / 1.409 = 0.809$ . If we assume that  $f(b)$  is not very sensitive to the dimensionality of the system we obtain  $\bar{d}_a(d=3) \approx 0.809 \times 1.722 \approx 1.39$ . If we use the same procedure for the animals we obtain  $d_a(d=3) \approx 1.895$ , only 5% smaller than the exact value  $d_a(d=3) = 2$  (Parisi and Sourlas 1981). Thus although this simple procedure is not expected to be too reliable, its estimates are in reasonable agreement with the available data. Hence for lattice animals we estimate

$$\bar{d}_a(d=2) \approx 1.12, \quad \bar{d}_a(d=3) \approx 1.37. \quad (9)$$

We are not aware of any data to compare with our estimates.

We have discovered an error in the calculation of Sahimi and Jerauld (1984) for  $d_w$  of lattice animals. The LHS of their equation (7) must be multiplied by a factor  $\frac{1}{2}$ . With this correction their PSRG method yields  $d_w(d=2) \approx 2.5$ . This is in reasonable agreement with the estimate of Wilke *et al* (1984) and Havlin *et al* (1984). If we use the Monte Carlo estimate of Wilke *et al* (1984),  $d_w(d=2) \approx 2.6 \pm 0.3$ , equation (4) would be satisfied. More recently Havlin *et al* (1984) estimated that  $d_w \approx 2.78 \pm 0.08$ . With this estimate equation (4) is almost satisfied. One should also expect that at any given dimension,  $d_w$  for lattice animals will be smaller than  $d_w$  for the largest percolation cluster at  $p_c$ .

In three dimensions our estimate of  $\bar{d}_a$  together with the exact result  $d_a = 2$  yield  $d_w(d=3) \approx 3.37$ , if we invoke equation (4). This is exactly the estimate of Havlin *et al* (1984) who obtained  $d_w(d=3) \approx 3.37 \pm 0.1$ . It is also consistent with the result of Wilke *et al* (1984) who estimated that  $d_w(d=3) \approx 3.4 \pm 0.4$ . Therefore, it appears that equation (4) may also be satisfied at  $d=3$  for lattice animals (as discussed above). It would be interesting to see whether the fractal dimension of backbone lattice animals is a monotonic function of dimensionality. If we assume that for lattice animals  $\beta_L = \bar{\beta}_L$  and use the estimates for  $d_s$ ,  $d_a$  and  $\bar{d}_a$ , we obtain  $\bar{d}_s(d=2) \approx 1$ . This is an interesting result in that  $\bar{d}_s$  cannot be less than unity and this result is just on the borderline. Thus, more work is necessary to settle this interesting issue.

Aharony and Stauffer (1984) have recently proposed that for a fractal with  $d_f \leq 2$  one has  $d_w = d_f + 1$ . This together with equation (4) means that  $\bar{d}_a = 1$  for  $d_a \leq 2$ . That is, the fractal dimensionality of the backbone of lattice animals is a constant for all  $1 \leq d \leq 3$ . This is an intriguing result which calls for more precise estimates of  $d_w$  and  $d_a$ . Note also that this relation means that  $d_2 = 2d_f/(1+d_f)$ .

We now consider DC conduction on lattice animals. To define the problem we assign a conductance  $g_0$  to every bond that joins two nearest-neighbour sites in the animal. We then expect that as the number  $N$  of elements in the animal becomes very large, the conductivity  $\sigma$  of the animal scale with  $N$  as

$$\sigma \sim N^{-t_a}. \quad (10)$$

Here  $t_a$  is an exponent which we expect to be universal. We also expect  $\sigma$  to be an intrinsic property of the animal, so that it should not matter how  $\sigma$  is measured. For example, it can be measured between two bars of constant potential. We first note that an equation which relates  $t_a$  to other exponents of the animal can be obtained. This is done by noting that the relation  $t = (d-2)\nu + \zeta$  for percolation clusters can be generalised to any fractal. For example, for lattice animals one has

$$t_a = (d-2)/d_a + 2/d_s - 1. \quad (11)$$

This equation is also valid for the ws aggregates with the proper use of the fractal and spectral dimension of the aggregates. Equation (11) can be used to find the mean-field value of  $t_a$  for lattice animals, i.e. its value for  $d \geq 8$ . Because  $d_s(d \geq 8) = 4/3$  (Sahimi and Jerauld 1984) and  $d_a(d \geq 8) = 4$ , one obtains

$$t_a(d \geq 8) = 2. \quad (12)$$

In two dimensions equation (11) yields  $t_a \approx 0.73$  if we use  $d_s \approx 1.15$  which is the estimate of Havlin *et al* (1984). Likewise, we find  $t_a \approx 1.19$  for three-dimensional animals. If one assumes that the AO conjecture is exact for  $d_a \geq 2$ , one obtains the exact results  $t_a(d=3) = 1$  and  $t_a(d=4) = 4/3$ . On the other hand the Aharony-Stauffer argument means that  $t_a(d=2) \approx 0.64$ , which appears to be too low and thus it may indicate that

this conjecture does not hold for animals. Thus the Aharony-Stauffer relation may just be a lower bound for  $t_a$  (an upper bound for  $d_s$ ). We note that equation (12) can be used to measure the spectral dimension  $d_s$  of animals if  $t_a$  can be measured accurately, since very accurate or exact values of  $d_a$  are already known and direct measurement of  $d_s$  is very difficult (Wilke *et al* 1984). One very accurate method is to combine the transfer-matrix method for lattice animals (Derrida and DeSeze 1982) and for conduction (Derrida and Vannimenus 1982).

#### 4. The Witten-Sander aggregates

We now turn our attention to the ws aggregates. In the ws model the initial state at time  $\theta = 1$  is a seed particle located at the centre of a large lattice. A second particle is released at a distance far from the seed particle at time  $\theta = 2$ . The particle then performs a random walk until it reaches a site adjacent to the seed site, where it stops its random walk and joins the cluster. The process of releasing the particles is continued until a large cluster of occupied sites is formed. The radius of the cluster  $R$  scales with  $N$ , the total number of particles in the cluster, as

$$R \sim N^{1/d_{ws}}, \tag{13}$$

where  $d_{ws}$  is the fractal dimension of the aggregate. Meakin (1983) found that for the ws aggregates  $d_{ws} \approx 5d/6$  for  $d = 2-6$ . Moreover, Witten and Sander (1983) argued that these aggregates do not have upper critical dimension so that  $d_{ws} \rightarrow \infty$  as  $d \rightarrow \infty$ .

Here we estimate the fractal dimension  $\bar{d}_{ws}$  of the backbone of the ws aggregates by a PSRG method. This PSRG method is similar to the one developed by Gould *et al* (1983) and yields good results for  $d_{ws}$  at  $d = 2$  with small cells. Certain difficulties arise when one employs this PSRG method to study ws aggregates in three and higher dimensional systems or when one uses very large RG cells, as has been discussed elsewhere (Sahimi and Jerauld 1983, Nakanishi and Family 1984). But for the present purpose the PSRG method of Gould *et al* is a useful tool.

The PSRG treatment of the backbone of the ws aggregates is a two-parameter model. A fugacity  $S_B$  is assigned to each occupied site of the backbone (which is defined in a way similar to the backbone sites of lattice animals), and another fugacity  $W$  is assigned to each step of the random walk of the added particle. Thus the recursion relation for  $S'_B$ , the renormalised backbone site occupation fugacity, is written as

$$S'_B = \sum_{s,m} C_{sm} S_B^s W^m,$$

where  $C_{sm}$  is the number of different ways of growing a spanning backbone of  $s$  sites generated by random walks with a total number of steps  $m$ . Another recursion relation is found for  $W'$ , the fugacity of a single-step random walk on the renormalised lattice. This relation is of the form

$$W' = \sum_n b_n W^n,$$

where  $b_n$  is the total number of walks of  $n$  steps that start from a fixed origin and span the cell in a given direction. Since the random walk is diffusive, i.e. the length of the walk  $l$  obeys  $l \sim N_w^{1/2}$ , where  $N_w$  is the number of steps of the walk, one may enumerate only those walks whose number of steps  $N_w$  obeys  $N_w \leq l^2$ , where  $l$  is



taken to be the end-to-end length of the walk on the cell because, at the critical fugacity  $W^*$ , only these walks are important (Gould *et al* 1983). For example, for a  $b = 2$  cell, all walks of up to five steps are enumerated, since  $l = 5^{1/2}$ . The fractal dimension  $\bar{d}_{ws}$  of the backbone of the aggregate is calculated by the usual way, i.e.  $\bar{d}_{ws} = \ln \lambda_{ws} / \ln b$ , where  $\lambda_{ws} = \partial S'_B / \partial S_B$  evaluated at the fixed points  $S_B = S_B^*$  and  $W = W^*$ . We used standard RG cells, see e.g. Gould *et al* (1983).

The fractal dimension  $\bar{d}_{ws}$  of the backbone of the ws aggregates were calculated for cell sizes  $b = 2, 3$ , and 4 on a square lattice; the results are  $\bar{d}_{ws} \approx 1.707, 1.52$ , and 1.48 respectively. These values extrapolate to a value  $\bar{d}_{ws} \approx 1.25$  if one plots the values of  $\bar{d}_{ws}(b)$  against  $1/\ln b$ . A similar PSRG calculation for the fractal dimension of the aggregates themselves yields  $d_{ws} \approx 1.67$  (Gould *et al* 1983), in complete agreement with the computer simulation results of Meakin (1983). These values for  $d_{ws}$  and  $\bar{d}_{ws}$  do not satisfy equation (4) since one then obtains,  $d_w(d = 2) \approx 2.92$ , in contrast with the estimate (Meakin and Stanley 1983),  $d_w(d = 2) \approx 2.56 \pm 0.10$ . On the other hand, the Aharony–Stauffer argument that  $d_w = 1 + d_{ws}$  is consistent with the result of Meakin and Stanley (1983) for  $d_w$ , which also implies that  $\bar{d}_{ws} = 1$ . However, this estimate for  $d_w$  is not accurate. More recent calculations (McKarnin, Nordahl, Sahimi and Tirrell, to be published) show that while  $d_s(d = 2) \approx 1.2$ , one has  $d_w(d \approx 2) \approx 2.76$ . Thus the Aharony–Stauffer relation does not hold for the ws aggregates. The importance of equation (4) is that it relates the dynamical exponent  $d_w$  to the static exponents in a simple manner. By using  $\bar{d}_{ws}(d = 2) \approx 1.25$ ,  $d_{ws}(d = 2) \approx 1.67$  and (Meakin and Stanley 1983)  $d_s(d = 2) \approx 1.20$  and the equality of  $\beta_L$  and  $\bar{\beta}_L$ , we obtain  $\bar{d}_s(d = 2) \approx 1.06$  for the ws aggregates.

Since for the ws aggregates  $d_{ws} \rightarrow \infty$  as  $d \rightarrow \infty$ , equation (11) predicts that  $t_{ws}(d \rightarrow \infty) = 2/d_s(d \rightarrow \infty)$ . The fact that  $d_{ws}(d \rightarrow \infty) \rightarrow \infty$  means that on a Bethe lattice, i.e. an infinite-dimensional structure, all sites of the lattice will be occupied by the incoming particles. Since on this lattice for any cluster of linear dimension  $L$  we expect the number of sites, which is proportional to the resistance of the cluster, to be proportional to  $L^2$ , we deduce that  $\beta_L = -2$  for the ws aggregates in the limit  $d \rightarrow \infty$  ( $d_{ws} \rightarrow \infty$ ). This result was also obtained by Family and Coniglio (1984). By invoking equation (3) we then obtain  $d_s(d \rightarrow \infty) = 2$  for the ws aggregates, in contrast with percolation clusters and lattice animals for which  $d_s = 4/3$  for this high-dimensionality limit. This indicates that the spectral dimension of the ws aggregates varies between a value of about (Meakin and Stanley 1983)  $d_s(d = 2) \approx 1.20$  and  $d_s(d \rightarrow \infty) = 2$ . This may mean that for kinetic clusters such as the ws aggregates one has, for the high-dimensionality limit,  $d_s = 2$ , whereas  $d_s = 4/3$  for ‘equilibrium’ clusters such as percolation clusters and lattice animals. Since all of the sites of a Bethe lattice will be occupied by the incoming particles, one cannot distinguish between the aggregate and its backbone; consequently  $\bar{d}_s(d \rightarrow \infty) = 2$  for the ws aggregates. Therefore, if we invoke (11), we obtain

$$t_{ws}(d \rightarrow \infty) = 1. \quad (14)$$

These results demonstrate clearly the difference between the ws aggregates and ‘equilibrium’ clusters. By using the numerical estimates of Meakin (1983) and Meakin and Stanley (1983) we obtain  $t_{ws}(d = 2) \approx 0.67$  and  $t_{ws}(d = 3) \approx 0.94$ . One may also estimate  $t_{ws}$  by an accurate method such as that of Turban and Debierre (1984) from which  $d_s$  for the ws aggregates can be calculated. But our results so far imply that the AO conjecture may not be exact at any dimension for the ws aggregates. On the other hand, it may also mean that the AO conjecture may hold for the ws aggregates for a

range of  $d$  such that  $d_l \leq d \leq d_u$ , where  $d_l \approx 2.4$  is the dimensionality at which  $d_{ws} = 2$ . For  $d > d_u$  one may expect that  $d_s$  increases from its values of  $d_s \approx 4/3$  to reach its high-dimensionality limit of 2. One may then identify  $d_u$  as a sort of upper critical dimensionality for the ws aggregates. An estimate of  $d_u$  would then be of interest.

## 5. Summary and conclusions

We have studied the properties of the backbone of, and conduction in, random clusters which have homogeneous interior structure. These include percolation clusters, lattice animals and the ws aggregates. For lattice animals the fractal dimension of the backbone was calculated, for the first time, with the help of which the hypothesis of Wilke *et al* (equation (4)) was tested for lattice animals; the results indicate that it may hold in low dimensions. The mean-field value of the critical exponent of conductivity for lattice animals was found to be 2, in contrast with that of the Witten-Sander aggregates which was shown to be 1. The spectral and fractal dimensions of the backbone of lattice animals and the ws aggregates were estimated at two dimensions and were found to be different from (and slightly lower than) the corresponding values for lattice animals and the ws aggregates themselves. It was shown that for the ws aggregates  $d_s(d \rightarrow \infty) = \bar{d}_s(d \rightarrow \infty) = 2$ , in contrast with  $d_s = 4/3$  and  $\bar{d}_s = 1$  for percolation clusters and lattice animals.

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